Moderate deviations for linear eigenvalue statistics of β-ensembles

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Outline

β-ensemble

- β-ensemble with the potential V: the distribution in $\mathbb{R}^N$ defined by

$$dP_{N,\beta}^{V;[b_-,b_+]}(x) = \frac{1}{Z_{N,\beta}^{V;[b_-,b_+]}} \exp\left\{-\frac{\beta}{2} H_N^{V;[b_-,b_+]}(x)\right\} I_{[b_-,b_+]^N}(x)dx. \tag{1.1}$$

- Here,

$$H_N^{V;[b_-,b_+]}(x) := \sum_{1\le i\neq j\le N} -\log|x_i - x_j| + N\sum_{i=1}^N V(x_i). \tag{1.2}$$

considered as a function of x, is the Hamiltonian.

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$$Z_{N,\beta}^{V;[b_-,b_+]} := \int_{\mathbb{R}^N} \exp\left\{-\frac{\beta}{2} H_N^{V;[b_-,b_+]}(x)\right\} I_{[b_-,b_+]^N}(x)dx \tag{1.3}$$

is the normalizing constant, called the partition function,

- The parameter β is a positive number which is proportional to the inverse of the absolute temperature
- The function $V : \mathbb{R} \cap [b_-, b_+] \to \mathbb{R}$, called the potential, is a continuous function.
- Assume that if $b_\tau = \tau\infty$ where $\tau = -$ or $+$, then V satisfies the growth condition

$$\liminf_{x \to \tau\infty} \frac{V(x)}{2 \log |x|} > 1. \tag{1.4}$$

- When $\beta = 1, 2, 4$, and $V(x) = x^2/2$, such distributions correspond to the eigenvalue distributions of real symmetric, Hermitian and symplectic matrix modles, respectively.

- The logarithmic potential energy functional on the probability measure space $\mathcal{P}([b_-, b_+])$ over $[b_-, b_+]$ defined by

$$I_V(\mu) = \int_{[b_-,b_+]\times[b_-,b_+]} \log|x-y|^{-1}\,\mu(dx)\,\mu(dy) + \int_{[b_-,b_+]} V(x)\,\mu(dx) \tag{1.5}$$

- The equilibrium measure is a unique global minimizer μ_V which has a compact support, denoted by Σ_V, and μ_V is characterized by the existence of a constant c_V such that

$$\begin{aligned} \zeta_V(x) &\geq c_V \text{ for all } x \in [b_-, b_+], \\ \zeta_V(x) &= c_V \text{ for } \mu_V\text{- almost surely } x, \end{aligned} \tag{1.6}$$

where

$$\zeta_V(x) := \int \log|x-y|^{-1}\mu_V(dy) + \frac{V(x)}{2}. \tag{1.7}$$

Empirical measure and linear statistics

- The *empirical measure* L_N is defined by:

$$L_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}. \tag{1.8}$$

- The *unnormalized empirical measure* M_N

$$M_N := \sum_{i=1}^{N} \delta_{x_i}. \tag{1.9}$$

- The linear statistics associated to real function g is defined by

$$M_N(g) := \int_{\mathbb{R}} g(x)\, M_N(dx). \tag{1.10}$$

Background

- LLN and LDP for the empirical measure: The empirical measure L_N converges almost surely to the *equilibrium measure*) μ_V, and satisfies a large deviation principle (see Ben Arous and Guionnet(1997)).
- CLT for the linear statistics: Johansson(1998), Shcherbina(2013), Borot and Guionnet(2013), Bekerman, Leblé and Serfaty(2018). See Bai and Silverstein (2010) for the linear statistics of eigenvalues of random matrices.
- Stein method and Berrey-Esseen bounds for the linear statistics: Lambert, Ledoux and Webb (2017).
- MDP for the empirical measure: Dembo, Guionnet and Zeitouni (2003) for Gaussian ensembles.
- **Our purpose**: To study MDP for the linear statistics of β-ensembles

Basic assumption (H1)

Regularity The function V and g are continuous functions over $[b_-, b_+]$.

Confinement If $b_\tau = \tau\infty$ is infinite, then the growth condition (1.4) holds.

One-cut regime The support Σ_V of μ_V consists in unique interval $[\alpha_-, \alpha_+] \subset [b_-, b_+]$.

Control of LD On $[b_-, b_+] \backslash (\alpha_-, \alpha_+)$, the function $\zeta_V(x)$ achieves its minimum value at α_- and α_+ only.

Offcriticality $S(x) > 0$ whenever $x \in [\alpha_-, \alpha_+]$, where

$$S(x) := \pi \frac{d\mu_V}{dx} \sqrt{\left| \frac{\prod_{\tau' \in \mathrm{Hard}} (x - \alpha_{\tau'})}{\prod_{\tau \in \mathrm{Soft}} (x - \alpha_\tau)} \right|}, \qquad (2.1)$$

and $\tau \in \mathrm{Hard}$ (resp. $\tau \in \mathrm{Soft}$) iff $\alpha_\tau = b_\tau$ (resp. $\tau(b_\tau - \alpha_\tau) > 0$).

Analyticity V and g can be extended as a holomorphic function on some open neighborhood U of $[\alpha_-, \alpha_+]$.

CLT

$M_N(g) - N\int g(\xi)\mu_V(d\xi)$ converges in law as $N \to \infty$ to a Gaussian distribution with mean

$$m(g) := \left(1 - \frac{2}{\beta}\right)\int \psi'(x)\mu_V(dx) \tag{2.2}$$

and variance

$$v(g) := -\frac{2}{\beta}\int \psi(x)g'(x)\mu_V(dx), \tag{2.3}$$

where the constant c_g and the function ψ of class C^2 in some open neighborhood U of $[\alpha_-, \alpha_+]$ are the solution of

$$-\frac{1}{2}\psi(x)V'(x) + \int \frac{\psi(x)-\psi(y)}{x-y}\mu_V(dy) = \frac{g(x)}{2} + c_g \text{ for } x \in U. \tag{2.4}$$

Main result

Theorem 2.1

Let r_N be a sequence of positive numbers satisfying

$$r_N \to \infty, \quad r_N/N \to 0.$$

Assume that (H1) holds. Then $\frac{1}{r_N}\left(\sum_{i=1}^N g(\lambda_i) - N\int g(\xi)\mu_V(d\xi) - m(g)\right)$ *satisfies a large deviation principle with speed* r_N^2 *and good rate function* $J(x) = \frac{x^2}{2v(g)}$, *that is, for any closed* $F \subset \mathbb{R}$,

$$\limsup_{N\to\infty} \frac{1}{r_N^2} \log P_{N,\beta}^{V;[b_-,b_+]}\left(\frac{\sum_{i=1}^N g(\lambda_i) - N\int g(\xi)\mu_V(d\xi) - m(g)}{r_N} \in F\right) \leq -\inf_{x\in F} J(x), \tag{2.5}$$

and for any open $F \subset \mathbb{R}$,

$$\liminf_{N\to\infty} \frac{1}{r_N^2} \log P_{N,\beta}^{V;[b_-,b_+]}\left(\frac{\sum_{i=1}^N g(\lambda_i) - N\int g(\xi)\mu_V(d\xi) - m(g)}{r_N} \in G\right) \geq -\inf_{x\in G} J(x). \tag{2.6}$$

It is sufficient to prove that for $t \in [-r_N, r_N]$ uniformly

$$\log E_{N,\beta}^{V}\left(\exp\{tM_N(g)\}\right)$$
$$= tN\int g(\xi)\mu_V(d\xi) + tm(g) + \frac{t^2}{2}v(g) + O\left(\frac{r_N^3}{N}\right), \qquad (2.7)$$

where $E_{N,\beta}^{V}$ means the expectation with the distribution $P_{N,\beta}^{V;[a_-,a_+]}$, and $[a_-, a_+]$ is a finite interval with

$$[\alpha_-, \alpha_+] \subset [a_-, a_+] \subset [b_-, b_+]$$

We can write

$$
\begin{aligned}
&\log E^V_{N,\beta}\left(\exp\left\{\frac{-\beta t}{2}M_N(g)\right\}\right)\\
&= -\frac{\beta}{2}\int_0^t E^{V^s_N}_{N,\beta} M_N(g)ds\\
&= -\frac{\beta}{2}\int_0^t \oint_{\mathcal{C}([a_-,a_+])} \frac{1}{2i\pi} g(\eta) W_1^{V^s_N}(\eta)d\eta ds
\end{aligned}
$$

where $\mathcal{C}([a_-, a_+])$ is a contour surrounding $[a_-, a_+]$ inside U,

$$
V^t_N(x) = V(x) + \frac{t}{N}g(x), \tag{2.8}
$$

and the correlator

$$
W_1^{V^t_N}(x) = E^{V^t_N}_{N,\beta}\left(\int \frac{M_N(d\xi)}{x-\xi}\right), \tag{2.9}
$$

Therefore, we need to give an expansion of the correlator $W_1 = W_1^{V_N^t}(x)$ such that

$$\int_0^t \oint_{\mathcal{C}([a_-,a_+])} \frac{1}{2i\pi} g(\eta) W_1^{V_N^s}(\eta) d\eta ds$$
$$= tN \int g(\xi)\mu_V(d\xi) + tm(g) + \frac{t^2\beta}{2} v(g) + O\left(\frac{r_N^3}{N}\right)$$

Representations of m(g) and v(g)

Set

$$\mathcal{H}_{1;[a_-,a_+]} = \{f : \text{holomorphic on } \mathbb{C} \setminus [a_-, a_+], f(x) = O(1/x), x \to \infty\}$$

and

$$\mathcal{H}_{2,[a_-,a_+]} = \left\{f \in \mathcal{H}_{1;[a_-,a_+]};\ f(x) = O(1/x^2) \text{ as } x \to \infty\right\}.$$

Define the bounded linear operator from $\mathcal{H}_{2;[a_-,a_+]}$ to $\mathcal{H}_{1;[a_-,a_+]}$:

$$(\mathcal{K}f)(x) := 2W_1^{\{-1\}}(x)f(x) - \oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c\right) V'(\xi)f(\xi) \tag{2.10}$$

where

$$W_1^{\{-1\}}(x) = \int \frac{\mu_V(d\xi)}{x-\xi}, \quad L(x) := \prod_{\tau \in \mathrm{Hard}} (x - \alpha_\tau) \tag{2.11}$$

and the constant

$$c = \begin{cases} 0, & \text{If Soft} = \{\pm\} \text{ or Hard} = \{\pm\}, \\ \frac{1}{a_\tau - a_{-\tau}}, & \text{if } \tau \in \text{ Soft and } -\tau \in \text{Hard}. \end{cases} \tag{2.12}$$

Then

$$(\mathcal{K}^{-1}f)(x) = \oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \frac{1}{\xi - x} \frac{\tilde{\sigma}(\xi)}{\tilde{\sigma}(x)} \frac{f(\xi)}{2Y(\xi)}, \tag{2.13}$$

where

$$Y(x) = -W_1^{\{-1\}}(x) + \frac{V'(x)}{2}, \quad \tilde{\sigma}(x) = \sqrt{(x-\alpha_-)(x-\alpha_+)}. \tag{2.14}$$

We can prove that $m(g)$ and $v(g)$ have the following representations.

$$m(g) = \oint_{\mathcal{C}([a_-,a_+])} \frac{1}{2i\pi} g(\eta) r(\eta) d\eta,$$

$$v(g) = -\frac{2}{\beta} \oint_{\mathcal{C}([a_-,a_+])} \frac{1}{2i\pi} g(\eta) \mathcal{L}(g)(\eta) d\eta.$$

where

$$\mathcal{L}(g)(x) := \mathcal{K}^{-1} \left[\oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c \right) g'(\xi) W_1^{\{-1\}}(\xi) \right], \tag{2.15}$$

$$r(x) := -\left(1 - \frac{2}{\beta}\right) \mathcal{K}^{-1} \left(\frac{d}{dx} W_1^{\{-1\}}(x) + \sum_{\tau \in \mathrm{Hard}} \frac{1}{(a_\tau - a_{-\tau})(x - a_\tau)} \right). \tag{2.16}$$

Thus, the key of the proof is to obtain the following expansion:

$$W_1(x) = NW_1^{\{-1\}}(x) + r(x) + t\mathcal{L}(g)(x) + o\left(\frac{r_N^2}{N}\right) \qquad (2.17)$$

since

$$\int_0^t \oint_{\mathcal{C}([a_-,a_+])} \frac{1}{2i\pi} g(\eta)\left(NW_1^{\{-1\}}(\eta) + r(\eta) + s\mathcal{L}(g)(\eta)\right)(\eta)d\eta ds$$

$$= tN\int g(\xi)\mu_V(d\xi) + tm(g) + \frac{t^2\beta}{2}v(g)$$

Loop equations

Generally, the *correlators* $W_n^{V_N^t}, n \geq 1$ are defined by

$$W_n(x_1, \cdots, x_n) := W_n^{V_N^t}(x_1, \cdots, x_n) = \partial_{\varepsilon_1} \cdots \partial_{\varepsilon_n} \log \left(Z_{N,\beta}^{V_N^t - \frac{2}{N\beta} \sum_{i=1}^n \frac{\varepsilon_i}{x_i - \cdot}} \right)_{\varepsilon_1 = \cdots = \varepsilon_n = 0} \quad (2.18)$$

They satisfy the following loop equations:

(1). For any $x \in \mathbb{C} \setminus [a_-, a_+]$,

$$
\begin{aligned}
& W_2(x,x) + (W_1(x))^2 + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx}(W_1(x)) \\
& + (N(1-2/\beta) - N^2) \sum_{\tau \in \mathrm{Hard}} \frac{1}{(a_\tau - a_{-\tau})(x - a_\tau)} \\
& - N \left(\oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c \right) \frac{dV_N^t(\xi)}{d\xi} W_1(\xi) \right) \\
& - \frac{2}{\beta} \sum_{\tau \in \mathrm{Soft}} \frac{\partial_{a_\tau} \log Z_N^{V_N^t;[a_-,a_+]}}{x - a_\tau} = 0
\end{aligned}
\tag{2.19}
$$

(2). For any $n \geq 2$, $x \in \mathbb{C} \setminus [a_-, a_+]$, $x_I = (x_i)_{i \in I} \in (\mathbb{C} \setminus [a_-, a_+])^{n-1}$,

$$
\begin{aligned}
& W_{n+1}(x, x, x_I) + \sum_{J \subset I} W_{|J|+1}(x, x_J) W_{n-|J|}(x, x_{I \setminus J}) + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx}(W_n(x, x_I)) \\
=& N \left(\oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x - \xi} + c \right) \frac{dV_N^t(\xi)}{d\xi} W_n(\xi, x_I) \right) \\
& - \frac{2}{\beta} \sum_{i \in I} \frac{d}{dx_i} \left(\frac{W_{n-1}(x, x_{I \setminus \{i\}})}{x - x_i} - \frac{L(x_i)}{L(x)} \left(\frac{1}{x - x_i} + c \right) W_{n-1}(x_I) \right) \\
& + \frac{2}{\beta} \sum_{\tau \in \text{Soft}} \frac{\partial_{a_\tau} W_{n-1}(x_I)}{x - a_\tau}.
\end{aligned}
\tag{2.20}
$$

Priori estimates for correlators

- For $t \in [-r_N, r_N]$ uniformly,

$$\|W_1 - NW_1^{\{-1\}}\|_\Gamma \leq C_{\Gamma,V,\|g\|,\|g'\|} w_N, \tag{2.21}$$

  where

$$w_N = \sqrt{N(\log N + r_N)}. \tag{2.22}$$

- Suppose $w_N \to \infty$ with $N \to \infty$ and $\|W_1 - NW_1^{\{-1\}}\|_\Gamma \leq C_{\Gamma,V,g} w_N$. Then for each $n \geq 2$, there exists a constant $C_n = C_{n,\Gamma,V,g}$ such that for $t \in [-r_N, r_N]$ uniformly,

$$\|W_n(x_1, \cdots, x_n)\|_\Gamma \leq C_n (w_N)^n \tag{2.23}$$

Proof of (2.17)

Set $\delta W_1 = W_1 - NW_1^{\{-1\}}$. Then by the priori estimates and the loop equations, we get

$$(\delta W_1)(x) = O(\sqrt{N(\log N + r_N)}) = o(N),$$

$$W_2(x,x) = O(N(\log N + r_N)) = o(N^2),$$

and

$$\frac{\partial_{a_\tau} \log Z_N^{V_N^t;[a_-,a_+]}}{x - a_\tau} = o(1)$$

Using the loop equations again, one can get

$$(\mathcal{K}\delta W_1)(x) + \frac{1}{N}((\delta W_1)(x))^2 = O(r_N),$$

$$\delta W_1 = O(r_N),$$

$$(\delta W_1)^2 = o(Nr_N),$$

and

$$(\mathcal{K}\delta W_1)(x) = t\oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi}\frac{L(\xi)}{L(x)}\left(\frac{1}{x-\xi}+c\right)g'(\xi)W_1^{\{-1\}}(\xi)+o(r_N).$$

Thus

$$\delta W_1(x) = t\mathcal{L}(g)(x) + o(r_N).$$

By the loop equations (2.20), we can obtain

$$\frac{\partial_{a_\tau} W_1(y)}{x - a_\tau} = o(1), \quad W_3(x,x,y) = O(r_N^3), \quad W_2(x,y) = O(r_N^2),$$

$$W_2(x,x) = O\left(\frac{r_N^3}{N} + 1\right) = o(r_N^2),$$

and

$$\begin{aligned}\mathcal{K}(\delta_2 W_1)(x) = &-\left(1 - \frac{2}{\beta}\right)\left(\frac{d}{dx} W_1^{\{-1\}}(x) + \sum_{\tau \in \mathrm{Hard}} \frac{1}{(a_\tau - a_{-\tau})(x - a_\tau)}\right) \\ &+ \frac{t^2}{N}\left(\oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c\right) g'(\xi)\mathcal{L}(g)(\xi) - (\mathcal{L}(g)(x))^2\right) \\ &+ o\left(r_N^2/N\right),\end{aligned}$$

where $\delta_2 W_1 = \delta_1 W_1 - t\mathcal{L}(g)$, Therefore, we have

$$\delta_2 W_1(x) = r(x) + \frac{t^2}{N}\tilde{\mathcal{L}}(g)(x) + o\left(\frac{r_N^2}{N}\right),$$

where

$$\tilde{\mathcal{L}}(g)(x) := \mathcal{K}^{-1}\left(\oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi}\frac{L(\xi)}{L(x)}\left(\frac{1}{x-\xi}+c\right)g'(\xi)\mathcal{L}(g)(\xi) - (\mathcal{L}(g)(x))^2\right).$$

That is,

$$W_1(x) = NW_1^{\{-1\}}(x) + r(x) + t\mathcal{L}(g)(x) + \frac{t^2}{N}\tilde{\mathcal{L}}(g)(x) + o\left(\frac{r_N^2}{N}\right).$$

Therefore, we have

$$\log E^V_{N,\beta}\left(\exp\left\{t\left(M_N(g)-N\int g(\xi)\mu_V(d\xi)-m(g)\right)-\frac{t^2}{2}v(g)\right\}\right)$$
$$=\frac{t^3}{N}u(g)+o\left(\frac{r_N^3}{N}\right),$$

where

$$u(g):=\frac{4}{3\beta^2}\oint_{\mathcal{C}([a_-,a_+])}\frac{1}{2i\pi}g(\eta)\tilde{\mathcal{L}}(g)(\eta)d\eta.$$

G. W. Anderson, A. Guionnet, O. Zeitouni.
An introduction to Random Matrices,
Cambridge Studies in Advanced Mathematics **118**, Cambridge University Press, 2010.

Z. D. Bai, J. W. Silverstein.
Spectral Analysis of Large Dimensional Random Matrices.
Second Edition, Springer New York, 2010.

F. Bekerman, T. Leblé, and S. Serfaty.
CLT for fluctuations of β-ensembles with general potential.
Electron. J. Probab. 23(115), 1–31, 2018.

G. Ben Arous and A. Guionnet.
Large deviations for Wigner's law and Voiculescu's non-commutative entropy.
Probab. Theory Related Fields, **108**, 517–542, 1997.

A.Borodin, V. Gorin and A.Guionnet.
Gaussian asymptotics of discrete β-ensembles,
Publications mathématiques de l'IHÉS, 125(1), 1–78, 2017.

G. Borot and A. Guionnet.
Asymptotic expansion of β matrix models in the one-cut regime.
Communications in Mathematical Physics, 317(2):447–483, 2013.

G. Borot and A. Guionnet, K. K. Kozlowski.
Large-N asymptotic expansion for mean field models with Coulomb gas interaction.
Int. Math. Res. Not. IMRN 2015(20), 10451-10524, 2015.

A. Dembo, A. Guionnet and O. Zeitouni.
Moderate deviations for the spectral measure of certain random matrices.
Ann. Inst. Henri Poincar, Probab. Stat., 39(6):1013–1042, 2003.

K. Johansson.
On fluctuations of eigenvalues of random Hermitian matrices.
Duke Mathematical Journal, 91(1):151–204, 1998.

G. Lambert, M. Ledoux, and C. Webb.
Stein's method for normal approximation of linear statistics of beta-ensembles.
https://arxiv.org/abs/1706.10251, 06 2017.

M. Shcherbina.
Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime.
Journal of Statistical Physics, 151(6):1004–1034, 2013.

Thank you!